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Determinantal forms for composite Schur and Q-functions via the boson–fermion correspondence

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Abstract. The boson–fermion correspondence is used to represent a composite Schur function as a fermionic expectation value. Wick's theorem then enables it to be written as a determinant, generalizing the Giambelli formula for ordinary Schur functions as single-hook determinants. The Schur Q-function analogue, recently introduced in the context of the BKP hierarchy, is also given.

1. Introduction

The algebra of free fermions has been used effectively in studying various problems in theoretical physics, chief amongst them the Ising model [1, 2] and the KP hierarchy and generalizations [3, 4]. In this latter work by the Kyoto School (see [5] for a review), the so-called boson–fermion correspondence was a major tool enabling, for instance, the role of affine Lie algebras as symmetries of integrable hierarchies to be clarified.

Recently, a method has been developed [6] to study the KP hierarchy which is rooted in symmetric function theory [7]. This method is based on the observation that the vertex operator underlying the hierarchy (and the related affine Lie algebra $gl(\infty)$) are intimately connected with Schur (S-) functions. An important ingredient is the fact that matrix elements of the vertex operator in an S-function basis are given by composite S-functions [8], which were introduced earlier in the context of finite-dimensional Lie algebras [9]. For instance, explicit evaluation of the composite S-function by means of a determinantal formula [10] led to a new proof of the well known fact that Schur polynomials solve the KP hierarchy [5].

The method of [6] can also be applied to the BKP hierarchy, with the role of S-functions this time played by Schur Q-functions. A particular Q-function series, with a similar structure to the composite S-function, played a similarly important role there and was called the composite Q-function. A determinantal formula was lacking at the time and left as an open problem. In a later paper [11], which produced a new realization of the affine Lie algebra $go(\infty)$ on the space of distinct partitions, explicit calculations of composite Q-functions were required. General results were conjectured on the basis of a number of small calculations based on the Q-function series definition. However, the result of [11] is but a generalization of that of [5] on the realization of $gl(\infty)$ on the space of all partitions. This latter result, in turn, was obtained using the boson–fermion correspondence. This

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thus strengthened the suspicion that the boson–fermion correspondence and the theory of composite S- and Q-functions are related.

In fact, the boson–fermion correspondence has recently been invoked [12] to prove certain S- and Q-function identities! In the present paper, we present the modification of [12] necessary to derive determinantal formulae for both composite S- and Q-functions. The result for the composite S-function is different from the ones already available [10, 13]. Apart from being useful in the method of [6], it is expected that this new determinantal formula has applications in the representation theory of finite-dimensional Lie algebras and superalgebras (the original setting for composite S-functions). On the other hand, the (Pfaffian) formula for the composite Q-function allows the results of [11] to be proved. More generally, the results here provide the link between the fermion-based techniques of [5] and the symmetric function theoretical method of [6].

2. Review of composite S- and Q-functions

2.1. S-functions

Let $x = (x_1, x_2, x_3, \ldots)$ be an infinite number of indeterminates and $\frac{1}{x}$ denote $(\frac{1}{x_1}, \frac{1}{x_2}, \ldots)$. Let P be the set of all partitions, with e.g. $\lambda \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n) \in P$ being a partition of $|\lambda| = \sum_{i=1}^{n} \lambda_i$ into $l(\lambda) \equiv n$ parts with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_n > 0$. Then for $\mu, \nu \in P$, the composite S-function $s_{\bar{\mu};\nu}(x)$ is defined [9] as

$$s_{\bar{\mu};\nu}(x) = \sum_{\xi \in P} (-1)^{|\xi|} s_{\nu/\xi}(x) s_{\mu/\xi'}(\tfrac{1}{x}) \tag{2.1}$$

where λ' denotes the partition conjugate to λ and $s_{\mu/\xi}(x)$ etc are skew S-functions defined in terms of S-functions $s_\lambda(x)$ by $s_{\mu/\xi}(x) = \sum_\lambda c^{\mu}_{\xi\,\lambda} s_\lambda(x)$, with $c^{\mu}_{\xi\,\lambda}$ being Littlewood–Richardson coefficients. An S-function in turn is defined as

$$s_\lambda(x) = \det\left(s_{(\lambda_i - i + j)}(x)\right)_{1 \leqslant i, j \leqslant l(\lambda)} \tag{2.2}$$

with $s_{(n)}(x) \equiv h_n(x)$ ($n \in \mathbb{Z}$) being complete symmetric functions whose generating function is

$$\sum_{m=0}^{\infty} h_m(x) t^m = \prod_{i=1}^{\infty} (1 - x_i t)^{-1}$$

for $n \geqslant 0$ and $s_{(n)}(x) = 0$ for $n < 0$. When $\mu = 0$ (the empty partition) the composite S-function $s_{\bar{\mu};\nu}(x)$ reduces to the ordinary S-function $s_\nu(x)$. The theory of symmetric functions can be found in [7], whereas [14] contains a survey of composite (and ordinary) S-functions with emphasis on applications to Lie (super) algebra representation theory.

It is a fundamental fact that the set of S-functions $\{s_\lambda(x)\}_{\lambda \in P}$ forms a basis for the ring $\Lambda(x)$ of symmetric functions of x. Furthermore an inner product $\langle\ ,\ \rangle$ can be defined on $\Lambda(x)$ such that the S-function basis is orthonormal: $\langle s_\mu(x), s_\nu(x)\rangle = \delta_{\mu\,\nu}$. If $D(f)$ for any symmetric function f denotes the adjoint with respect to $\langle\ ,\ \rangle$ of multiplication by f, then the skew function $s_{\mu/\xi}(x)$ admits an alternative interpretation as $s_{\mu/\xi}(x) = D(s_\xi(x)) s_\mu(x)$. Another important basis for $\Lambda(x)$ is given by the set of power sum symmetric functions $p_\lambda(x) \equiv p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots$, with $p_m(x) = x_1^m + x_2^m + \cdots$. The power sum basis is orthogonal with respect to the same inner product and [7]

$$D(p_n(x)) = n \frac{\partial}{\partial p_n(x)}. \tag{2.3}$$

In transforming between the S-function and power sum bases, the following relations are useful:

$$\begin{aligned}\sum_{\lambda} s_\lambda(x)s_\lambda(y) &= \exp\left(\sum_{m=1}^{\infty} \tfrac{1}{m} p_m(x)p_m(y)\right) \\ \sum_{\lambda}(-1)^{|\lambda|} s_\lambda(x)s_{\lambda'}(y) &= \exp\left(-\sum_{m=1}^{\infty} \tfrac{1}{m} p_m(x)p_m(y)\right).\end{aligned} \tag{2.4}$$

Define the S-function 'supersymmetric' in the variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ to be [15]

$$s_\lambda(x/y) = \sum_{\mu \in P} (-1)^{|\mu|} s_{\lambda/\mu}(x)s_{\mu'}(y). \tag{2.5}$$

Then $\{s_\lambda(x/y)\}_{\lambda \in P}$ forms a basis for the ring $\Lambda(x/y)$ in which multiplication is defined in exactly the same way as in $\Lambda(x)$, namely in terms of the Littlewood–Richardson coefficients. Furthermore, the relations (2.4) also hold with the arguments x and y replaced by x/w and y/z respectively, if we define the supersymmetric power sum $p_n(x/y)$ to be $p_0(x/y) = 1$ and $p_n(x/y) = p_n(x) - p_n(y)$ for $n > 1$. With this in mind, all the subsequent results for S-functions in $\Lambda(x)$ are equally valid for supersymmetric S-functions in $\Lambda(x/y)$.

As shown in [8] composite S-functions can be expressed in the form

$$s_{\bar{\mu};\nu}(z) = (-1)^{|\mu|}\langle s_\nu(x), \Gamma(z)s_{\mu'}(x)\rangle \tag{2.6}$$

where

$$\Gamma(z) = \exp\left(\sum_{k=1}^{\infty} \tfrac{1}{k} p_k(z)p_k(x)\right)\exp\left(-\sum_{k=1}^{\infty} \tfrac{1}{k} p_k(\tfrac{1}{z})D(p_k(x))\right) \tag{2.7}$$

and the inner product is with respect to the 'reference' space $\Lambda(x)$. Due to (2.3) we can identify

$$\begin{aligned} H_k &\leftrightarrow D(p_k(x)) \qquad k > 0 \\ H_{-k} &\leftrightarrow p_k(x) \qquad k \geqslant 0 \end{aligned} \tag{2.8}$$

where H_k, $k \in \mathbb{Z}$, generates the Heisenberg algebra $[H_m, H_n] = m\delta_{m+n,0}$. In a later section, this will be seen to be a key ingredient in the boson–fermion correspondence. The operator $\Gamma(z)$ is then identified with

$$V(z) = \exp\left(\sum_{k=1}^{\infty} \tfrac{1}{k} p_k(z)H_{-k}\right)\exp\left(-\sum_{k=1}^{\infty} \tfrac{1}{k} p_k(\tfrac{1}{z})H_k\right). \tag{2.9}$$

If $z = (z_1, z_2, \ldots)$ is specialized to $z_1 = z$, $0 = z_2 = z_3 = \cdots$ (with abuse of notation) then $V(z)$ can be recognized as the simplest kind of vertex operator, involved in, for example, certain aspects of the KP hierarchy [5]. More generally, the specialization $z_1 = z_2 = \cdots = z_\alpha = z$, $0 = z_{\alpha+1} = z_{\alpha+2} = \cdots$ (formally, for α not a positive integer) results in the vertex operator

$$V_\alpha(z) = \exp\left(\alpha\sum_{k=1}^{\infty} \tfrac{1}{k} z^k H_{-k}\right)\exp\left(-\alpha\sum_{k=1}^{\infty} \tfrac{1}{k} z^{-k} H_k\right)$$

familiar from string theory. On the other hand, the 'supersymmetric' case $V(z/w)$, where $z = (z_1, z_2, \cdots)$ and $w = (w_1, w_2, \cdots)$ are specialized to $z_1 = z, 0 = z_2 = z_3 = \cdots$ and $w_1 = w, 0 = w_2 = w_3 = \cdots$, is the vertex operator involved in the definition of the affine Lie algebra $gl(\infty)$ and also in the tau function bilinear identity of the KP hierarchy. If we now identify $p_\lambda(y)$ as the boson Fock state $H_{-\lambda_1}H_{-\lambda_2}\cdots|0\rangle$, with $H_n|0\rangle = 0$ for $n > 0$, then the S-functions form an alternative basis for the boson Fock space and the composite S-function (2.6) has the interpretation as a vertex operator matrix element in the S-function basis.

2.2. *Q-functions*

Schur Q-functions were first introduced in the theory of projective representations of the symmetric group. They play the role of S-functions for the space $\Lambda_B(x)$ spanned by power sums $p_n(x)$ with odd n. Firstly, let $\mathrm{Pf}(A)$ for any antisymmetric matrix $(A_{ij})_{1\leqslant i,j\leqslant n}$ of even size n denote its Pfaffian defined inductively as

$$\mathrm{Pf}(A) = A_{12}$$

for $n = 2$, and for $n > 2$

$$\mathrm{Pf}(A) = A_{12}\,\mathrm{Pf}(A^{(12)}) - A_{13}\,\mathrm{Pf}(A^{(13)}) + \cdots + A_{1n}\,\mathrm{Pf}(A^{(1n)})$$

where $A^{(ij)}$ is the antisymmetric submatrix of A obtained by deleting the ith, jth rows and ith, jth columns. $\mathrm{Pf}(A)$ is one of the square roots of $\det(A)$. In the next two sections, we will need to use the following two properties of Pfaffians [12]:

(i) If A and X are $n \times n$ complex matrices and A is antisymmetric, then

$$\mathrm{Pf}(XAX^{\mathrm{T}}) = \det(X)\,\mathrm{Pf}(A)\,. \tag{2.10}$$

(ii) If B is an $n \times n$ non-singular complex matrix, then

$$\mathrm{Pf}\begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & 0 \end{pmatrix} = (-1)^{n(n-1)/2}\det(B)\,. \tag{2.11}$$

Q-functions are then defined as follows: For $m, n \geqslant 0$ define $Q_m(x)$ through the generating function

$$\sum_{m=0}^{\infty} Q_m(x)t^m = \prod_{i=1}^{\infty}\left(\frac{1+x_it}{1-x_it}\right)$$

and $Q_{mn}(x)$ by

$$Q_{mn}(x) = Q_m(x)Q_n(x) + 2\sum_{j=1}^{n}(-1)^j Q_{m+j}(x)Q_{n-j}(x)\,.$$

Note that $Q_{mn}(x) = -Q_{nm}(x)$ and $Q_{m0}(x) = Q_m(x)$. If $m, n < 0$ then set $Q_m(x) = Q_{mn}(x) = 0$. For any $\lambda = (\lambda_1, \ldots, \lambda_n) \in DP$, the set of partitions with distinct parts, the Q-function is defined as

$$Q_\lambda(x) = \mathrm{Pf}\big(Q_{\tilde\lambda_i\tilde\lambda_j}(x)\big) \tag{2.12}$$

where $\tilde\lambda = \lambda$ for n even and $\tilde\lambda = (\lambda_1, \ldots, \lambda_n, 0)$ for n odd.

The set of Q-functions $\{Q_\lambda(x)\}_{\lambda\in DP}$ forms a basis for $\Lambda_B(x)$. There exists an inner product $\langle\ ,\ \rangle_B$ with respect to which Q-functions are orthogonal

$$\langle Q_\lambda(x), Q_\mu(x)\rangle_B = b_\lambda\delta_{\lambda\mu}$$

where $b_\lambda = 2^{l(\lambda)}$. If $D(f)$ denotes the adjoint with respect to $\langle\ ,\ \rangle_B$ of multiplication by f, then skew Q-functions are defined by $Q_{\lambda/\mu}(x) = b_\mu^{-1}D(Q_\mu(x))Q_\lambda(x)$. Furthermore, $D(p_n(x)) = (n/2)\partial/\partial p_n(x)$ for n odd with respect to this inner product. Note the factor of two difference between this and (2.3).

In the study of [6] on the KP hierarchy, composite S-functions arose in the form (2.6). The corresponding object for the BKP hierarchy had S-functions replaced by Q-functions and $\Gamma(z)$ replaced by

$$\Gamma_B(z) = \exp\left(\sum_{n \text{ odd}} \tfrac{2}{n} p_n(z) p_n(x)\right) \exp\left(-\sum_{n \text{ odd}} \tfrac{2}{n} p_n(\tfrac{1}{z}) D(p_n(x))\right). \tag{2.13}$$

This can be evaluated using the relation

$$\sum_{\lambda \in DP} b_\lambda^{-1} Q_\lambda(x) Q_\lambda(y) = \exp\left(\sum_{n \text{ odd}} \tfrac{2}{n} p_n(x) p_n(y)\right)$$

to obtain

$$\langle Q_\nu(x), \Gamma_B(z) Q_\mu(x)\rangle_B = (-1)^{|\mu|} Q_{\bar{\mu};\nu}(z) \tag{2.14}$$

where the composite Q-function is defined as

$$Q_{\bar{\mu};\nu}(x) = \sum_{\xi \in DP} (-1)^{|\xi|} b_\xi Q_{\nu/\xi}(x) Q_{\mu/\xi}(\tfrac{1}{x}). \tag{2.15}$$

(This definition differs from that in [6] by a factor b_μ.)

The theory of S-functions is well known to be connected with the combinatorics of Young tableaux [7]. Similarly, composite S-functions are connected with composite Young tableaux [9] or rational tableaux [16], while Q-functions can be related to the theory of shifted tableaux [17]. It would be interesting to see if composite Q-functions have a meaning in this context.

3. The boson–fermion correspondence

3.1. Free fermions

The algebra of (charged) free fermions [3] is given by $\left(\bigoplus_{i\in\mathbb{Z}} \mathbb{C}\psi_i\right) \oplus \left(\bigoplus_{i\in\mathbb{Z}} \mathbb{C}\psi_i^*\right)$ with defining relations $\{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0$, $\{\psi_i, \psi_j^*\} = \delta_{ij}$. The Fock representation $\mathcal{F}$ is defined by the choice of vacuum $|0\rangle$ such that (different from [5], same as [12]): $\psi_i|0\rangle = 0$ for $j \leqslant 0$, $\psi_j^*|0\rangle = 0$ for $j > 0$. An inner product (,) exists with respect to which the states $\psi_{i_1} \cdots \psi_{j_1} \cdots |0\rangle$ with $i_k > 0$, $j_k \leqslant 0$ are orthonormal and ψ_j and ψ_j^* are adjoint to each other. In other words, the vacuum expectation value (defined as $\langle a\rangle \equiv (|0\rangle, a|0\rangle)$ for $a \in \mathcal{F}$) of a product of n free fermions is given by

$$\langle \psi_i \psi_j^*\rangle = \begin{cases} \delta_{ij} & j \leqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \psi_i^* \psi_j\rangle = \begin{cases} \delta_{ij} & j > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \psi_i \psi_j\rangle = \langle \psi_i^* \psi_j^*\rangle = 0$$

for the case when $n = 2$ and the free fermions are either just ψ_i or ψ_i^*. More generally we have Wick's theorem:

$$\langle w_1 w_2 \cdots w_n\rangle = \begin{cases} \mathrm{Pf}\left(\langle w_i w_j\rangle\right) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{3.1}$$

for any free fermions w_i. Define the operator H_n by $H_n = \sum_{i\in\mathbb{Z}} :\psi_i\psi^*_{i+n}:$, where $: :$ stands for normal ordering. Then H_n acts on the free fermions as a shift operator, namely $[H_n, \psi_i] = \psi_{i-n}$, $[H_n, \psi^*_i] = -\psi^*_{i+n}$, and furthermore $H_n|0\rangle = 0 = \langle 0|H_{-n}$ for $n > 0$. H_n also generate the Heisenberg algebra.

Introduce now the operator $\mathrm{e}^{H(x)}$ with $H(x) = \sum_{n=1}^{\infty} \frac{1}{n} p_n(x) H_n$. Then for $n \geqslant 0, m > 0$ we have

$$\langle \mathrm{e}^{H(x)}\psi_m\psi^*_{-n}\rangle = (-1)^n s_{(m,1^n)}(x)\,. \tag{3.2}$$

This is proved by making use of relations like

$$\mathrm{e}^{H(x)}\psi(z)\mathrm{e}^{-H(x)} = \psi(z)\exp\left(\sum_{n=1}^{\infty}\tfrac{1}{n}p_n(x)z^n\right) \tag{3.3}$$

where $\psi(z) = \sum_{i\in\mathbb{Z}}\psi_i z^i$, to write the left-hand side of (3.2) as

$$\sum_{k=0}^{n}(-1)^k s_{(m+n-k)}(x)s_{(1^k)}(x)$$

which is then recognized as the expansion along the first row of the determinantal expression of the right-hand side of (3.2) according to (2.2).

There exists another determinantal definition of S-functions (due to Giambelli), being [7]

$$s_\lambda(x) = \det\left(s_{(\lambda_i-i+1,1^{\lambda'_j-j})}(x)\right)_{1\leqslant i,j\leqslant r(\lambda)} \tag{3.4}$$

where $r(\lambda)$ is the Frobenius rank of λ, being the number of boxes along the main diagonal of its corresponding Young diagram. Note that $\{\lambda_i - i\}_{i=1,\dots,r(\lambda)}$ and $\{\lambda'_i - i\}_{i=1,\dots,r(\lambda)}$ are respectively the arm and leg lengths of the Young diagram and each is a positive and strictly decreasing set. Let $a(\lambda) = \sum_{i=1}^{r(\lambda)}(\lambda_i - i)$ and $b(\lambda) = \sum_{i=1}^{r(\lambda)}(\lambda'_i - i)$ be respectively the number of boxes above and below the main diagonal of the Young diagram for λ. Now, on using (3.2), $s_\lambda(x)$ can be written as a determinant whose (i, j)th entry is given by the fermionic expectation value $(-1)^{\lambda'_j-j}\langle \mathrm{e}^{H(x)}\psi_{\lambda_i-i+1}\psi^*_{-\lambda'_j+j}\rangle$. Using a special form of Wick's theorem (which is implied by (3.1) upon application of (2.11)), given by

$$\langle\Psi_1\cdots\Psi_n\Psi^*_n\cdots\Psi^*_1\rangle = \det\left(\langle\Psi_i\Psi^*_j\rangle\right)$$

and valid for free fermions of the form $\Psi_i = \sum a_{ij}\psi_j$, $\Psi^*_i = \sum b_{ij}\psi^*_j$, the S-function can thus be represented in the form [5, 12]

$$s_\lambda(x) = (-1)^{b(\lambda)+r(\lambda)(r(\lambda)-1)/2}\langle \mathrm{e}^{H(x)}\psi_{\lambda_1}\cdots\psi_{\lambda_{r(\lambda)}-r(\lambda)+1}\psi^*_{-\lambda'_1+1}\cdots\psi^*_{-\lambda'_{r(\lambda)}+r(\lambda)}\rangle\,. \tag{3.5}$$

Note that in arriving at (3.5), we have made use of the fact that the vacuum expectation on its right-hand side can be written as $\langle\left(\mathrm{e}^{H(x)}\psi_{\lambda_1}\mathrm{e}^{-H(x)}\right)\left(\mathrm{e}^{H(x)}\psi_{\lambda_2}\mathrm{e}^{-H(x)}\right)\cdots\rangle$, which is valid since $\mathrm{e}^{-H(x)}|0\rangle = |0\rangle$.

Equation (3.5) demonstrates explicitly the isomorphism of the map σ from $\mathcal{F}^{(0)}$, the charge-zero sector of the fermionic Fock representation, to $\Lambda(x)$ given by [3]

$$\sigma\left(a|0\rangle\right) = \langle \mathrm{e}^{H(x)}a\rangle \tag{3.6}$$

for a given $a|0\rangle$ in $\mathcal{F}^{(0)}$. This is known as the boson–fermion correspondence. The Heisenberg generators H_n (which constitute the bosons) can be seen to act on $\Lambda(x)$ via (cf (2.8))

$$\sigma H_k\sigma^{-1} = D(p_k(x)) \qquad k > 0$$

$$\sigma H_{-k}\sigma^{-1} = p_k(x) \qquad k \geqslant 0\,.$$

Furthermore, the inner products on $\Lambda(x)$ and $\mathcal{F}^{(0)}$ can be shown to be related by

$$\langle \sigma\,(a|0\rangle)\,,\sigma\,(b|0\rangle)\rangle = (a|0\rangle, b|0\rangle) \tag{3.7}$$

for any $a|0\rangle$ and $b|0\rangle$ in $\mathcal{F}^{(0)}$. Finally, we note the relation $\sigma V(z)\sigma^{-1} = \Gamma(z)$ which we will use later in deriving a determinantal formula for composite S-functions.

3.2. *Neutral free fermions*

We turn now to the algebra of free neutral fermions [4], given by $\oplus_{i\in\mathbb{Z}}\mathbb{C}\phi_i$ with defining relations $\{\phi_m, \phi_n\} = (-1)^m\delta_{m+n,0}$. The Fock representation $\mathcal{F}_B$ is defined with the choice of vacuum $\phi_i|0\rangle$ for $i < 0$. An inner product $(\ ,\ )_B$ exists with respect to which $\phi_{i_1}\cdots|0\rangle$ with $i_k \geqslant 0$ is orthogonal and ϕ_j is adjoint to $(-1)^j\phi_{-j}$. The vacuum expectation value of a product of n neutral free fermions is then given by

$$\langle\phi_i\phi_j\rangle = \begin{cases} (-1)^i\delta_{i+j,0} & i < 0 \\ 0 & i > 0 \\ \frac{1}{2}\delta_{j,0} & i = 0 \end{cases}$$

for the case $n = 2$ and more generally by Wick's theorem of the same form as (3.1). Define the operator H_n^B by $H_n^B = \frac{1}{2}\sum_{i\in\mathbb{Z}}(-1)^{i-1}\phi_i\phi_{-n-i}$. Then we have $[H_n^B, \phi_i] = \phi_{i-n}$ and $H_n|0\rangle = 0 = \langle 0|H_{-n}$ for $n > 0$. H_n can also be shown to generate the Heisenberg algebra $[H_m, H_n] = \frac{n}{2}\delta_{m+n,0}$.

Introduce now the operator $\mathrm{e}^{H_B(x)}$ with $H_B(x) = \sum_{n\ \text{odd}}\frac{2}{n}p_n(x)H_n^B$. On using the relation

$$\mathrm{e}^{H_B(x)}\phi(z)\mathrm{e}^{-H_B(x)} = \phi(z)\exp\left(\sum_{n\ \text{odd}}\frac{2}{n}p_n(x)z^n\right)$$

where $\phi(z) = \sum_{i\in\mathbb{Z}}\phi_i z^i$, we have the following result: For $m, n \geqslant 0$,

$$\langle \mathrm{e}^{H_B(x)}\phi_m\phi_n\rangle = \tfrac{1}{2}Q_{mn}(x)\,. \tag{3.8}$$

Comparison with (2.12) leads to a representation of $Q_\lambda(x)$ as a Pfaffian of a matrix whose (i, j)th element is $2\langle \mathrm{e}^{H(x)}\phi_{\tilde{\lambda}_i}\phi_{\tilde{\lambda}_j}\rangle$. On application of Wick's theorem, we arrive at [18]

$$Q_\lambda(x) = 2^{l(\tilde{\lambda})/2}\langle \mathrm{e}^{H_B(x)}\phi_{\tilde{\lambda}_1}\phi_{\tilde{\lambda}_2}\cdots\phi_{\tilde{\lambda}_{l(\tilde{\lambda})}}\rangle \tag{3.9}$$

where $l(\tilde{\lambda}) = l(\lambda)$ $(l(\lambda)+1)$ if $l(\lambda)$ is even (odd).

Equation (3.9) shows the isomorphism of the map σ_B from $\mathcal{F}_B^+$, the subspace of $\mathcal{F}_B$ containing states $\phi_{i_1}\cdots\phi_{i_n}|0\rangle$ with n even, to $\Lambda_B(x)$ given by [4, 18]

$$\sigma_B\,(a|0\rangle) = \langle \mathrm{e}^{H_B(x)}a\rangle \tag{3.10}$$

for $a|0\rangle \in \mathcal{F}_B^+$. The whole Fock representation $\mathcal{F}_B$ is isomorphic to two copies of $\Lambda_B(x)$. One can show that the Heisenberg operators H_n^B act on $\Lambda_B(x)$ via

$$\sigma_B H_n\sigma_B^{-1} = D(p_n(x))$$

$$\sigma_B H_{-n}\sigma^{-1} = p_n(x)$$

for n positive and odd. The inner products on $\Lambda_B(x)$ and $\mathcal{F}_B^+$ can also be shown to be related by

$$\langle\sigma_B(a|0\rangle), \sigma_B(b|0\rangle)\rangle_B = (a|0\rangle, b|0\rangle)_B \tag{3.11}$$

for $a|0\rangle$ and $b|0\rangle$ belonging to $\mathcal{F}_B^+$. Finally, referring back to (2.13), we have $\sigma_B V_B(z)\sigma_B^{-1}$ where the vertex operator $V_B(z)$ is defined as

$$V_B(z) = \exp\Big(\sum_{n\ \mathrm{odd}} \tfrac{2}{n} p_n(z) H_{-n}\Big) \exp\Big(-\sum_{n\ \mathrm{odd}} \tfrac{2}{n} p_n(\tfrac{1}{z}) H_n\Big).$$

4. Determinantal forms

4.1. Composite S-functions

We now derive a determinantal formula for the composite S-function. From (2.6) and the boson–fermion correspondence (equations (3.6), (3.5) and (3.7)), we obtain

$$\begin{aligned} s_{\bar{\mu};\nu}(z) &= (-1)^{|\mu|}\left(\sigma^{-1}(s_\nu(x)), V(z)\sigma^{-1}(s_{\mu'}(x))\right) \\ &= (-1)^{|\mu|}(-1)^{a(\mu)+n(n-1)/2}(-1)^{b(\nu)+m(m-1)/2} \\ &\quad \times\langle \psi_{-\nu'_1+1}\cdots\psi_{-\nu'_m+m}\psi^*_{\nu_1}\cdots\psi^*_{\nu_m-m+1}V(z) \\ &\quad \times\psi_{\mu'_1}\cdots\psi_{\mu'_n-n+1}\psi^*_{-\mu_1+1}\cdots\psi^*_{-\mu_n+n}\rangle \end{aligned} \tag{4.1}$$

where we have set $n = r(\mu)$, $m = r(\nu)$. The fermionic expectation value in (4.1) can be written, on applying Wick's theorem, as

$$\mathrm{Pf}\begin{pmatrix} 0 & A & 0 & B \\ -A^{\mathrm{T}} & 0 & C & 0 \\ 0 & -C^{\mathrm{T}} & 0 & D \\ -B^{\mathrm{T}} & 0 & -D^{\mathrm{T}} & 0 \end{pmatrix} \tag{4.2}$$

where A, B, C, D are the matrices

$$A = \left(\langle \psi_{-\nu'_i+i}\psi^*_{\nu_j-j+1}V^{(-)}(z)\rangle\right)_{1\leqslant i,j\leqslant m}$$

$$B = \left(\langle \psi_{-\nu'_i+i}V(z)\psi^*_{-\mu_j+j}\rangle\right)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$$

$$C = \left(\langle \psi^*_{\nu_i-i+1}V(z)\psi_{\mu'_j-j+1}\rangle\right)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$$

$$D = \left(\langle V^{(+)}(z)\psi_{\mu'_i-i+1}\psi^*_{-\mu_j+j}\rangle\right)_{1\leqslant i,j\leqslant n}$$

and $V^{(\pm)}(z)$ have been defined to be $V^{(-)}(z) = \exp\big(\sum_{k=1}^{\infty}\frac{1}{k}p_k(z)H_{-k}\big)$ and $V^{(+)}(z) = \exp\big(-\sum_{k=1}^{\infty}\frac{1}{k}p_k(\frac{1}{z})H_k\big)$ so that $V^{(-)}(z)V^{(+)}(z) = V(z)$. The Pfaffian (4.2), on performing row and column transformations and using (2.10) and (2.11), reduces to

$$(-1)^{mn}(-1)^{(m+n)(m+n-1)/2}\det\begin{pmatrix} A & B \\ -C^{\mathrm{T}} & D \end{pmatrix}. \tag{4.3}$$

Using the results

$$\langle V^{(+)}(z)\psi_p\psi^*_{-q}\rangle = (-1)^p s_{(q+1,1^{p-1})}(\tfrac{1}{z}) \qquad (q\geqslant 0, p>0)$$

$$\langle \psi_{-p}\psi^*_q V^{(-)}(z)\rangle = (-1)^p s_{(q,1^p)}(z) \qquad (q>0, p\geqslant 0)$$

$$\langle \psi_{-p}V(z)\psi^*_{-q}\rangle = (-1)^p s_{\overline{(q)};(1^p)}(z) \qquad (q\geqslant 0, p\geqslant 0)$$

$$\langle \psi^*_p V(z)\psi_q\rangle = (-1)^{q-1} s_{\overline{(p-1)};(1^{q-1})}(\tfrac{1}{z}) \qquad (q>0, p>0)$$

which are proved in a similar manner to (3.2), to evaluate the entries in the matrices A, B, C, D, the determinant in (4.3) (upon transposition) becomes

$$(-1)^{b(\nu)}(-1)^{b(\mu)+r(\mu)} \det \begin{pmatrix} s_{(\nu_i-i+1,1^{\nu'_j-j})}(z) & s_{\overline{(\nu_i-i)};(1^{\mu'_j-j})}(\frac{1}{z}) \\ s_{\overline{(\mu_i-i)};(1^{\nu'_j-j})}(z) & s_{(\mu_i-i+1,1^{\mu'_j-j})}(\frac{1}{z}) \end{pmatrix}. \tag{4.4}$$

In getting from (4.1) to (4.4), all the signs cancel out (upon recognizing that $b(\mu)+a(\mu)+r(\mu)=|\mu|$) and the determinant in (4.4) itself can be rearranged to give the final result:

$$s_{\bar{\mu};\nu}(z) = \det \begin{pmatrix} s_{(\mu_i-i+1,1^{\mu'_j-j})}(\frac{1}{z}) & s_{\overline{(\mu_i-i)};(1^{\nu'_j-j})}(z) \\ s_{\overline{(\nu_i-i)};(1^{\mu'_j-j})}(\frac{1}{z}) & s_{(\nu_i-i+1,1^{\nu'_j-j})}(z) \end{pmatrix}. \tag{4.5}$$

The top left and bottom right blocks are simply the Giambelli matrices for S-functions. It is thus clear that the special cases $s_{\bar{0};\nu}(z) = s_\nu(z)$ and $s_{\bar{\mu};0}(z) = s_\mu(\frac{1}{z})$ hold. So too does the condition $s_{\bar{\mu};\nu}(z) = s_{\bar{\nu};\mu}(\frac{1}{z})$.

The determinant (4.5) for a composite S-function appears to be new. There exists two other determinants for $s_{\bar{\mu};\nu}(z)$, the first being [10]

$$s_{\bar{\mu};\nu}(z) = (-1)^{m(m-1)/2} \det\Big(s_{\overline{(1^{\mu'_j-j+m})};(\nu_i-i+m)}(z)\Big) \tag{4.6}$$

and used in [6] to prove that Schur polynomials solve the KP hierarchy. Here, $m \geqslant \max(\mu_1, \mu'_1, \nu_1, \nu'_1)$ and $\nu_i = 0$ for $i > l(\nu)$. The determinant (4.6) (with z specialized to N variables, N arbitrary) was derived by making use of yet another determinant form for an S-function (due to Foulkes)

$$s_\lambda(z) = \det\big(s_{(\lambda_i-i+1,1^{m-j})}(z)\big) \tag{4.7}$$

where $m \geqslant \max(\lambda_1, \lambda'_1)$, and relating them by means of a known relation between $U(N)$ characters of which both $s_\lambda(z)$ and $s_{\bar{\mu};\nu}(z)$ can be so considered. In (4.7), $s_{(p,1^q)}(z)$ for $p \leqslant 0$ has to be interpreted as being $(-1)^q \delta_{p+q,0}$. In fact, this is consistent with (3.2) since ψ_p can then be moved past ψ_{-q} to annihilate the vacuum. Hence, the Foulkes determinant (4.7) leads us, by exactly the same arguments as for the Giambelli determinant, to an alternative representation of the S-function as a fermionic expectation value:

$$s_\lambda(z) = \langle e^{H(x)} \psi_{\lambda_1} \psi_{\lambda_2} \cdots \psi_{\lambda_m - m+1} \psi^*_{-m+1} \psi^*_{-m+2} \cdots \psi^*_0 \rangle. \tag{4.8}$$

It is conceivable, but not altogether obvious, that using this representation instead of (3.5) and going through the analogous steps to those for deriving (4.5) will lead us instead to (4.6).

Both (4.5) and (4.6) can be considered as 'reduced' determinantal forms, being respectively of rank $r(\mu)+r(\nu)$ and $\max(\mu_1, \mu'_1, \nu_1, \nu'_1)$. For completeness, we present the third determinantal form known for $s_{\bar{\mu},\nu}(z)$, given by [13, 19]

$$s_{\bar{\mu},\nu}(z) = \det \begin{pmatrix} s_{(1^{\mu'_l+k-l})}(\frac{1}{x}) & s_{(1^{\nu'_j-k-j+1})}(x) \\ s_{(1^{\mu'_l-i-l+1})}(\frac{1}{x}) & s_{(1^{\nu'_j+i-j})}(x) \end{pmatrix} \tag{4.9}$$

where $1 \leqslant i, j \leqslant \nu_1, 1 \leqslant k, l \leqslant \mu_1$ and i, j, k, l have to be read from top to bottom, left to right, bottom to top and right to left, respectively. A Laplace expansion of the determinant in (4.9) can be seen to be just the S-function series in the definition (2.1). For typical partitions μ and ν, the form (4.5) is the most economical and (4.9) the least.

4.2. Composite Q-functions

By (2.14) and the equations of the boson–fermion correspondence ((3.9), (3.10) and (3.11)), the composite Q-function can be written in the form

$$\begin{aligned} Q_{\bar{\mu};\nu}(z) &= (-1)^{|\mu|}\left(\sigma_B^{-1}(Q_\nu(x)), V_B(z)\sigma_B^{-1}(Q_\mu(x))\right) \\ &= (-1)^{|\mu|}2^{l(\bar{\nu})/2+l(\bar{\mu})/2}(-1)^{|\nu|}(-1)^{l(\bar{\nu})(l(\bar{\nu})-1)/2} \\ &\quad\times\langle\phi_{-\bar{\nu}_1}\cdots\phi_{-\bar{\nu}_{l(\bar{\nu})}}V_B(z)\phi_{\bar{\mu}_1}\cdots\phi_{\bar{\mu}_{l(\bar{\mu})}}\rangle\,. \end{aligned} \tag{4.10}$$

By applying Wick's theorem, the fermionic expectation value in (4.10) can be written as the Pfaffian

$$\mathrm{Pf}\begin{pmatrix} E & F \\ -F^{\mathrm{T}} & G \end{pmatrix} \tag{4.11}$$

where the matrices E, F, G are given by

$$E = \left(\langle\phi_{-\bar{\nu}_i}\phi_{-\bar{\nu}_j}V_B^{(-)}(z)\rangle\right)_{1\leqslant i,j\leqslant l(\bar{\nu})}$$

$$F = \left(\langle\phi_{-\bar{\nu}_i}V_B(z)\phi_{\bar{\mu}_j}\rangle\right)_{1\leqslant i\leqslant l(\bar{\nu}),1\leqslant j\leqslant l(\bar{\mu})}$$

$$G = \left(\langle V_B^{(+)}(z)\phi_{\bar{\mu}_i}\phi_{\bar{\mu}_j}\rangle\right)_{1\leqslant i,j\leqslant l(\bar{\mu})}$$

where $V_B^{(-)}(z) = \exp\left(\sum_{n\ \mathrm{odd}}\frac{2}{n}p_n(z)H_{-n}\right)$ and $V_B^{(+)}(z) = \exp\left(-\sum_{n\ \mathrm{odd}}\frac{2}{n}p_n(\frac{1}{z})H_n\right)$. Using the results

$$\langle\phi_{-m}\phi_{-n}V_B^{(-)}(z)\rangle = (-1)^{m+n}\tfrac{1}{2}Q_{nm}(z)$$

$$\langle V_B^{(+)}(z)\phi_m\phi_n\rangle = (-1)^{m+n}\tfrac{1}{2}Q_{mn}(\tfrac{1}{z})$$

$$\langle\phi_{-m}V_B(z)\phi_n\rangle = (-1)^{m+n}\tfrac{1}{2}Q_{\overline{(m)};(n)}(\tfrac{1}{z})$$

(for $m, n \geqslant 0$) and (2.10) the Pfaffian (4.11) can be written as

$$\det\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\mathrm{Pf}\begin{pmatrix} Q_{\bar{\nu}_j\bar{\nu}_i}(z) & Q_{\overline{(\bar{\nu}_i)};(\bar{\mu}_j)}(\frac{1}{z}) \\ -Q_{\overline{(\bar{\mu}_i)};(\bar{\nu}_j)}(z) & Q_{\bar{\mu}_i\bar{\mu}_j}(\frac{1}{z}) \end{pmatrix} \tag{4.12}$$

where A and B are diagonal matrices with entries $A_{ij} = ((-1)^{\bar{\nu}_i}/\sqrt{2})\delta_{ij}$ and $B_{ij} = ((-1)^{\bar{\mu}_i}/\sqrt{2})\delta_{ij}$ respectively. Upon resolving the sign changes and the factors of $\sqrt{2}$ one finds that the composite Q-function becomes

$$Q_{\bar{\mu};\nu}(z) = (-1)^{l(\bar{\nu})(l(\bar{\nu})-1)/2}\,\mathrm{Pf}\begin{pmatrix} -Q_{\bar{\nu}_i\bar{\nu}_j}(z) & Q_{\overline{(\bar{\nu}_i)};(\bar{\mu}_j)}(\frac{1}{z}) \\ -Q_{\overline{(\bar{\mu}_i)};(\bar{\nu}_j)}(z) & Q_{\bar{\mu}_i\bar{\mu}_j}(\frac{1}{z}) \end{pmatrix}.$$

Alternatively, the sign factor can be absorbed into the Pfaffian to obtain

$$Q_{\bar{\mu};\nu}(z) = \mathrm{Pf}\begin{pmatrix} Q_{\bar{\nu}_i\bar{\nu}_j}(z) & \mathrm{i}Q_{\overline{(\bar{\nu}_i)};(\bar{\mu}_j)}(\frac{1}{z}) \\ -\mathrm{i}Q_{\overline{(\bar{\mu}_i)};(\bar{\nu}_j)}(z) & Q_{\bar{\mu}_i\bar{\mu}_j}(\frac{1}{z}) \end{pmatrix}. \tag{4.13}$$

The similarity in form between the matrices in (4.5) and (4.13) is striking. The top left and bottom right blocks in (4.13) are just matrices through which Q-functions are defined. It is apparent that the special cases $Q_{\bar{0};\nu}(z) = Q_\nu(z)$, $Q_{\bar{\mu};0}(z) = Q_\mu(\frac{1}{z})$ hold. To see that the condition $Q_{\bar{\mu};\nu}(z) = Q_{\bar{\nu};\mu}(\frac{1}{z})$ also holds, it is sufficient to note that

$$\mathrm{Pf}\begin{pmatrix} A & B \\ -B^{\mathrm{T}} & C \end{pmatrix} = \det\begin{pmatrix} -I_m & 0 \\ 0 & I_n \end{pmatrix}\mathrm{Pf}\begin{pmatrix} A & -B \\ B^{\mathrm{T}} & C \end{pmatrix}$$

for any pair of square matrices A and C of size m and n respectively; and if m is even (and by necessity also n) then the determinant is 1.

5. Conclusion

In this paper we have obtained new determinantal forms for composite S- and Q-functions by making use of the boson–fermion correspondence along the lines of [12]. Although this work was principally motivated by the wish to relate the formalism of [6] for the KP hierarchy to that of [3], it has thrown up a bonus regarding possible applications to the representation theory of finite-dimensional Lie (super) groups and algebras. To conclude, we name but two instances where determinantal formulae for composite S-functions have been useful: (i) equation (4.6) was used to derive a Robinson's hook length-type dimension formula for mixed tensor irreducible representations of $U(N)$ [20]. (ii) In studying supercharacters of $U(M/N)$ associated with composite S-functions, equation (4.9) was used to derive a modification rule used to express 'non-standard' supercharacters in terms of 'standard' supercharacters [19], required for instance in the interpretation of branching rules. The new determinantal form (4.5) can likewise be expected to be of use.

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